$$A_5=-I/Rt,\quad B_6=-(3-\nu)I/2R^2t,\quad C_5=1/R^2$$

$$A_6=\frac{(1-\nu)(RT-t)}{2\,Rt},\quad B_7=-\frac{2\omega^2\rho I(1-\nu)}{E\,R^2t}$$

$$A_7=-\frac{\omega^2\rho I(1-\nu^2)}{E\,R\,t},\quad C_7=-\frac{(1-\nu^2)\omega^2\rho}{E}$$

$$C_8=-I/Rt,\quad C_9=-(3-\nu)I/2R^2t,\quad C_{10}=\frac{Q(1-\nu)}{2\,R\,t}$$

$$C_{11}=-I\omega^2\rho(1-\nu^2)/ERt,$$

$$C_{12}=-2I\omega^2\rho(1-\nu^2)/ER^2t$$

$$C_{13}=\omega^2\rho I(1-\nu^2)/Et,\quad C_{14}=\frac{2Q}{R^2t}+\frac{I\omega^2\rho(1-\nu^2)}{R^2t\,E}$$

where

$$I=t^3/12$$

$$T=\log(1+t/2R)/(1-t/2R)$$

$$P=Rt[1+(t/2R)^2]$$

$$Q=(RT-t)$$

References

[1]Kraus, H., *Thin Elastic Shells*, Wiley, New York, 1967, p. 297.

Properties of Axial or Torsional Free-Vibration Frequency of Rods

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Introduction

It is well known that the free-vibration frequencies of a system with a finite number of degrees of freedom are given by the roots of the equation obtained by requiring the determinant of the dynamic matrix to vanish.[1] In the case of longitudinal or torsional vibrations of rods approximated by finite elements, the dynamic matrix is tridiagonal and possesses a repetitive structure, which enables one to show that the eigenvalues have certain interesting properties. These properties were observed by Johnson[2] while investigating minimum-weight rods under dynamic loads.

It will be shown that for a clamped rod with an odd number of degrees of freedom, the middle frequency is independent of any nonuniformity in the area distribution. Then it will be shown that for any rod, the frequencies in the lower half of its spectrum are conjugate to the frequencies in the upper half in the sense that if the design is modified in a way such as to maintain unchanged some frequency in the lower half spectrum, its conjugate frequency in the upper half will also remain unchanged, and finally, upper bounds on the highest frequency of both the lower half and entire spectra will be given.

For brevity, a rod with an off number of degrees of freedom (that is, a rod represented by an odd number of finite elements, Fig. 1) will be called an "odd rod," and an "even rod," is one with an even number of degrees of freedom. Although the analysis is carried out for longitudinal vibrations, it can be applied to torsional vibrations the same way.

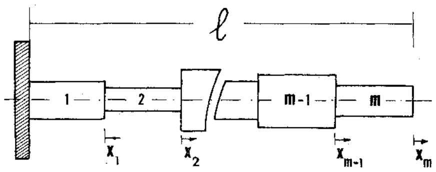

Fig. 1 Discretization of the rod in *m* finite elements of equal length.

Equations of Motion of Vibrating Rod

The element stiffness and inertia matrices can be written, respectively, as[1]

$$[K_i]=\frac{E}{\ell_i}\begin{bmatrix}A_i & -A_i\\ -A_i & A_i\end{bmatrix}$$

$$[I_i]=\frac{\rho\ell_i}{6}\begin{bmatrix}2A_i & A_i\\ A_i & 2A_i\end{bmatrix}\tag{1}$$

where: A_i = cross sectional area of i-th element; ℓ_i = length of i-th element; E = Young's modulus; and ρ = specific mass. If the rod is divided into m elements of equal length ℓ/m, with the degrees of freedom numbered sequentially as in Fig. 1, and the left end is assumed fixed, the equations of motion for harmonic oscillations with frequency ω reduce to

$$\begin{bmatrix}(A_1+A_2)\delta_1 & -A_2\delta_2 & & & & \\ -A_2\delta_2 & (A_2+A_3)\delta_1 & -A_3\delta_2 & & & \\ & & (A_3+A_4)\delta_1 & -A_4\delta_2 & & \\ & & & -A_{m-1}\delta_2 & (A_{m-1}+A_m)\delta_1 & -A_m\delta_2 \\ & & & & -A_m\delta_2 & A_m\delta_1\end{bmatrix}\{x\}=0\tag{2}$$

where: $\{x\}$ is a vector of axial displacements, and

$$\delta_1=1-(\omega^2\rho\ell^2/3Em^2)\qquad \delta_2=1+(\omega^2\rho\ell^2/6Em^2)$$

Property of Middle Frequency Invariance and Conjugate Pairs

The condition for the existence of nontrivial solution of Eq.

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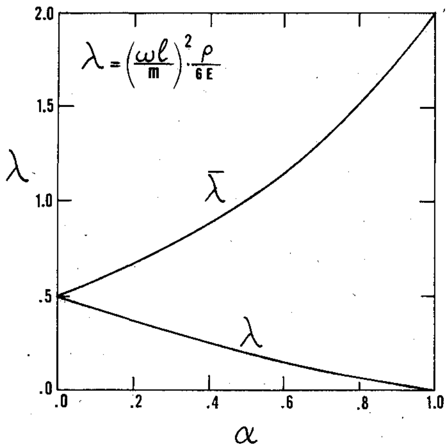

Fig. 2 Conjugate frequencies as function of α.

(2) is $D_m = 0$. But

$$D_m = (A_1 + A_2)\delta_1 D_{m-1} - A_2^2\delta_2^2 D_{m-2} \tag{3}$$

where: D_m is the determinant of Eq. (2), which has order m; D_{m-1} is the determinant of the matrix obtained by eliminating the first row and first column; and D_{m-2} is the determinant of the matrix obtained by eliminating the first two rows and columns. Based on the recursive relation (3), we can derive the following results: a) the middle frequency of an odd rod remains unchanged with respect to changes in the cross sectional areas; and b) conjugacy of the remaining frequencies. In order to prove these results, we prove the following lemmas.

Lemma a

If m is odd, δ_1 can be factored out of D_m.

Proof. Let it be assumed true for $m-2$. Then,

$$D_{m-2} = \delta_1 D'_{m-2} \tag{4}$$

Recalling relation (3), we observe that

$$D_m = \delta_1[(A_1 + A_2)D_{m-1} - A_2^2\delta_2^2 D'_{m-2}] \tag{5}$$

The result is, therefore, also true for m, which is seen to be odd since $m-2$ is odd. But $D_1 = A_1\delta_1$, hence the proof by induction is complete. Now it becomes simple to prove the initial statement. In fact, if m is odd, the equation $D_m = 0$ admits the solution $\delta_1 = 0$ since, according to lemma a, δ_1 can be factored out. Consequently, the frequency $\omega = (m/\ell)(3E/\rho)^{1/2}$ does not depend on any particular design. That this frequency is actually the middle one is a consequence of result b.

Lemma b

D_m is dimensionally homogeneous in δ_1 and δ_2. Moreover, δ_1 and δ_2 have only even powers, with the exception of the common factor δ_1 in case m is odd.

Proof: let the statement be true for D_{m-1} and D_{m-2}. We consider two cases: 1) $m-2$ is odd: recalling Eq. (3), we can write

$$D_m = \delta_1[(A_1 + A_2)D_{m-1} - A_2^2\delta_2^2 D'_{m-2}] \tag{6}$$

Since the assertion is assumed true for m-1 and $m-2$, D_{m-1} and D'_{m-2} have only even powers in δ_1 and δ_2. Except for the common factor δ_1, therefore, D_m will have only even powers in δ_1 and δ_2 too.

2) $m-2$ is even: once again, recalling Eq. (3), we can write

$$D_m = (A_1 + A_2)\delta_1{}^2 D'_{m-1} - A_2^2\delta_2^2 D_{m-2} \tag{7}$$

Using the same argument as item (1), we conclude that D_m has only even powers in δ_1 and δ_2. Since $D_1 = A_1\delta_1$ and $D_2 = (A_1 + A_2)\delta_1{}^2 A_2 - A_2^2\delta_2^2$, the proof is complete.

Consider now, the characteristic equation $D_m = 0$. We can divide both sides by $\delta_2^m \geq 1$ and eliminate the root $\delta_1 = 0$ in case m is odd. In view of lemma b, the characteristic equation can be written as

$$a_r V^r + a_{r-2}V^{r-2} + \ldots + a_0 = 0 \tag{8}$$

where r is even and $V = \delta_1/\delta_2$.

It becomes clear from the structure of Eq. (8) that if α is a root of the equation, so is $-\alpha$. Recalling the definitions of δ_1 and δ_2, we have

$$\omega_j^2 = \frac{6Em^2}{\rho\ell^2}\left(\frac{1-\alpha}{2+\alpha}\right)$$
$$\bar{\omega}_j^2 = \frac{6Em^2}{\rho\ell^2}\left(\frac{1+\alpha}{2-\alpha}\right) \tag{9}$$

As one can see, for every value of α there are two frequencies interrelated by Eqs. (9). We define conjugate frequencies as the pair related by Eqs. (9). Figure 2 shows the two branches given in Eqs. (9) as a function of α. One can see that if m is odd, and therefore the invariant frequency $\omega = (m/\ell)(3E/\rho)^{1/2}$ exists, half the remaining frequencies will be lower and the other half higher. Hence $\omega = (m/\ell)(3E/\rho)^{1/2}$ is the middle one. Finally, the following immediate results can also be stated: 1) the upper bound on the highest frequency is 2 $(m/\ell)(3E/\rho)^{1/2}$; and 2) the lower half range of frequencies if upper-bounded by $(m/\ell)(3E/\rho)^{1/2}$.

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Effects of Density and Velocity Ratio on Discrete Hole Film Cooling

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Nomenclature

- D = injection hole diameter
- I = momentum flux ratio ($=\rho_j U_j{}^2/\rho_\infty U_\infty{}^2$)
- M = mass flux ratio ($=\rho_j U_j/\rho_\infty U_\infty$)
- U = velocity
- X = distance downstream from row of holes
- ρ = density
- η = effectiveness (= local surface mass fraction of foreign gas/mass fraction of foreign gas in injectant)

Subscripts

- j = denotes the jet
- ∞ = denotes the mainstream

Introduction

CURRENT practice in the film cooling of gas turbine blades involves the use of rows of discrete holes for coolant injection. Much of the previous research on

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